Phase Transition of Electrically Charged Ricci-flat Black Holes

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Abstract

We study phase transition between electrically charged Ricci-flat black holes and AdS soliton spacetime of Horowitz and Myers in five dimensions. Boundary topology for both of them is $\mathbf{S^1} \times \mathbf{S^1} \times \mathbf{R^2}$. We consider AdS-Reissner-Nordström black hole and R-charged black holes and find that phase transition of these black holes to AdS soliton spacetime depends on the relative size of two boundary circles. We also perform the stability analysis for these black holes. In order to use the AdS/CFT correspondence, we work in the grand canonical ensemble.

Contents

1 Introduction

The correspondence between supergravity in anti-de Sitter(AdS) spacetime and conformal field theory in one lower dimension is one of the interesting subjects of current research. It has been conjectured [1] by Maldacena that the type IIB supergravity (superstring theory) on $\mathbf{AdS_5} \times \mathbf{S^5}$ is dual to the the $\mathcal{N} = 4$ super Yang-Mills theory living on the boundary of the AdS space. The conjecture also relates thermodynamics of the gauge theory on conformal boundary of the AdS space with that of the gravity residing in the bulk AdS.

    We will first review thermodynamics of the gauge theory on a three sphere $\mathbf{S^3}$. To compute the free energy or entropy of the finite temperature gauge theory, we have to first calculate the partition function on $\mathbf{S^3} \times \mathbf{S^1}$, where $\mathbf{S^1}$ is the Euclidean time circle. The circumference β of $\mathbf{S^1}$ is related to the inverse temperature of the field theory and we denote the radius of $\mathbf{S^3}$ as β'. Due to the conformal invariance of the gauge theory on $\mathbf{S^3} \times \mathbf{S^1}$, only the ratio of these two parameters β/β' is relevant.

Phase transition is one of the important aspects in the study of thermodynamics. A system in the finite volume, in general, does not exhibit any phase transition. But in the large N limit, *i.e.*, when the number of degrees of freedom goes to infinity then it is possible to have a phase transition even in finite volume [2]. It has been argued in [3] that the $\mathcal{N} = 4$ SYM theory on $\mathbf{S^3} \times \mathbf{S^1}$ shows a phase transition as a function of β/β' in large N limit. The large β/β' or small temperature phase corresponds to the confining phase where small β/β', *i.e.*, the large temperature phase corresponds to de-confining phase of the gauge theory.

On the gravity side Hawking and Page had shown in [4] that there exists a phase transition (HP phase transition) between spherical AdS (Schwarzschild) black hole and and global AdS spacetime. Above the transition temperature (Hawking-Page transition temperature), the black hole spacetime is stable while below that temperature the black hole spacetime is unstable and decays to the global AdS space time.

Using the AdS/CFT correspondence, Witten has identified the Hawking-Page phase transition in gravity side with the confinement-deconfinement phase transition in the gauge theory [3, 5]. The low temperature phase on the gravity side which is dominated by global AdS spacetime corresponds to the confining phase of gauge theory while the high temperature phase where AdS black hole is energetically favored corresponds to the deconfining phase of the gauge theory.

The phase transition, both on the gravity side and on the gauge theory side, is sensitive to the topology of spacetime. For example, instead of spherical asymptotic geometry if one considers AdS black holes with planar asymptotic geometry then it is easy to show that there exists no HP transition between this black hole phase and the global AdS spacetime. In other words, the planar black hole phase is always dominant for any non zero temperature. On the other hand dual thermal gauge theory of this planar black hole, which is defined on $\mathbf{S^1} \times \mathbf{R^3}$ also does not show any phase transition. The gauge theory is always in deconfined phase. This can be understood from the fact that planar geometry can be obtained from spherical geometry in $\beta' \to \infty$ limit and in this limit $\beta/\beta' \to 0$. The obvious question to ask here is, what happens when the black hole spacetime has asymptotic topology other than $\mathbf{S^1} \times \mathbf{S^3}$ or $\mathbf{S^1} \times \mathbf{R^3}$? We will try to address this question in the context of charged black holes in this paper.

To answer this question we have to first find out what kind of black hole topologies one can have in asymptotically AdS space. In fact, it is possible to construct a black hole solution of Einstein equation with positive (spherical), zero (flat) or negative (hyperboloid) constant curvature horizons in AdS spacetime [6]. Due to different horizon topologies, the

thermodynamic properties of these black holes are different. We will be focusing on black holes with flat horizon. It has been shown in [7, 8] that there are no phase transitions between the AdS black hole with Ricci flat horizon and the zero mass black hole (global AdS spacetime) [1] [2].

However, in [12], the authors have shown that it is possible to find a phase transition between Ricci-flat black holes and a AdS soliton spacetime where both the spacetime has asymptotic geometry is $\mathbf{S^1} \times \mathbf{S^1} \times \mathbf{R^2}$ ($\mathbf{S^1} \times \mathbf{R^2}$ is the Ricci-flat space) [3]. It has been conjectured by Horowitz and Myers [13] that given the Ricci-flat boundary topology $\mathbf{S^1} \times \mathbf{S^1} \times \mathbf{R^2}$ the AdS soliton spacetime is the minimum energy (perturbatively stable) solution of the Einstein equation. We will review briefly this conjecture of Horowitz and Myers and the phase transition between neutral black holes and AdS soliton spacetime in section 2.

Our main focus in this paper will be to generalize this analysis to charged black holes. Thermodynamics of charged black holes typically has more interesting features than the neutral ones. Lower dimensional charged black holes in the AdS space can arise in the compactification of string theory on $\mathbf{S^{n+1}}$. One simple way to get a charged black hole solution in lower dimensions (say, in 5 dimensions) is by compactifying the string theory (type IIB) on a rotating sphere (*i.e.*, $\mathbf{S^5}$ in this case)[22, 20]. In this paper, we will discuss the thermodynamics of Ricci flat charged black holes and phase transition between the black hole and AdS soliton spacetime.

Organization of our paper is as follows. In section 2 we briefly review the AdS soliton spacetime (section 2.1) and phase transition of neutral Ricci flat black holes (section 2.2). In the next section (section 3) we discuss the thermodynamics of charged Ricci-flat black holes in the grand canonical ensemble and study phase transition into AdS soliton spacetime. We show that the phase transition line depends on the size of the spatial $\mathbf{S^1}$ circle. We also discuss the thermodynamic stability of these black holes in section 4. The last section 5 contains discussion of our results. The detailed calculation leading to the on-shell Euclidean action for general R-charged black holes is given in the appendix A.

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[1]It is worth mentioning here that in studying the thermodynamics of Euclidean black hole a proper background subtraction is essential to get finite thermodynamic variables. One also has to ensure that the boundary topology of black hole spacetime and that of the background spacetime are same asymptotically (see [9] for an explicit discussion). For example, in asymptotically AdS space if we want to study the thermodynamics of AdS-Schwarzschildblack hole then the background we choose is asymptotically pure AdS spacetime.

[2]see [10] for related discussion in de-Sitter space and [11] for topological black holes with hair.

[3]Higher derivative correction to the phase transition of Ricci flat neutral black hole has been studied in [14] and in a recent paper [15].

2 AdS Soliton Spacetime and Phase Transition of Ricci-flat Black Holes

In this section we briefly review the work of Horowitz and Myers [13] and discuss how asymptotically Ricci flat black hole undergoes a phase transition into the AdS soliton spacetime [12].

2.1 AdS Soliton Spacetime

The AdS soliton spacetime is given by the metric,

$$ds_S^2 = -\frac{r^2}{b^2}dt_S^2 + \frac{b^2}{r^2}\left(1 - \frac{\mu^4}{r^4}\right)^{-1}dr^2 + \frac{r^2}{b^2}\left(1 - \frac{\mu^4}{r^4}\right)d\theta_S^2 + \frac{r^2}{b^2}h_{ij}dx^i dx^j, \quad (2.1)$$

where, b is radius of the AdS spacetime, μ is a constant parameter related to the energy of this spacetime and h_{ij} is the metric on a two dimensional Ricci-flat manifold R^2/Γ, where Γ is a finite discrete group. This two dimensional manifold $\mathbf{Y^2}$ can be a torus $\mathbf{T^2}$ for some non-trivial Γ or in the simpler case it is $\mathbf{R^2}$. We will consider $\mathbf{Y^2}$ to be $\mathbf{R^2}$ throughout this paper. This metric can be obtained as a solution of the Einstein equation (which is obtained by varying the Einstein-Hilbert action with negative cosmological constant). The metric can also be obtained by a double analytic continuation of a five dimensional Ricci-flat AdS black hole metric with $t \to i\theta_S$ and $\theta \to it_S$ (see [16] for related discussion),

$$ds^2 = -\frac{r^2}{b^2}\left(1 - \frac{\mu^4}{r^4}\right)dt^2 + \frac{b^2}{r^2}\left(1 - \frac{\mu^4}{r^4}\right)^{-1}dr^2 + \frac{r^2}{b^2}d\theta^2 + \frac{r^2}{b^2}h_{ij}dx^i dx^j. \quad (2.2)$$

In (2.1) the coordinate r is restricted to $r \geq \mu$ and θ_S must be identified with period,

$$\eta_S = \frac{\pi b^2}{\mu} \quad (2.3)$$

so as to avoid the conical singularity at $r = r_0$. The soliton spacetime X_S has a topology $\mathbf{R} \times \mathcal{B}^2 \times \mathbf{Y^2}$, where $\mathcal{B}^2$ is a two dimensional ball and $\mathbf{Y^2}$ is either $\mathbf{T^2}$ or $\mathbf{R^2}$ as mentioned above. Boundary M_S of this soliton spacetime has the topology of $\mathbf{R} \times \mathbf{S^1} \times \mathbf{Y^2}$.

The AdS soliton spacetime has negative energy and it is given by,

$$E_S = -\frac{Vb^3\pi^3}{16\pi G_5\eta_S^3}, \quad (2.4)$$

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where $V = \int \sqrt{h} d^2 x$. We can write this energy in terms of gauge theory variables using the AdS/CFT dictionary,

$$b^4 = \frac{N\sqrt{2G_{10}}}{\pi^2},$$
$$G_5 = \frac{G_{10}}{Vol(S^5)} \tag{2.5}$$

as,

$$E_S = -\frac{V\pi^2 N^2}{8\eta_S^3}. \tag{2.6}$$

The energy density of this spacetime then becomes,

$$\rho_S = \frac{E}{V\eta_S} = -\frac{\pi^2 N^2}{8\eta_S^4}. \tag{2.7}$$

Using the AdS/CFT correspondence we can compare the energy density of this soliton spacetime with the ground state energy of the dual field theory on $\mathbf{S^1 \times Y^2}$ where the length of S^1 is $\eta_{\mathbf{S}}$. Here the field theory is $\mathcal{N}{=}4$ SYM theory with SU(N) gauge group. The fermions are antiperiodic on $\mathbf{S^1}$. To compare the result obtained on the gravity side, we need to determine the Casimir energy of the weakly coupled gauge theory on the boundary of AdS soliton spacetime with the same boundary condition for fermions in $\mathbf{S^1}$ direction. When $\mathbf{Y^2 = R^2}$, the leading order result for the Casimir energy density is given by[4] [17],

$$\rho_{gauge} = -\frac{\pi^2 N^2}{6\eta_S^4}. \tag{2.8}$$

Thus the negative energy density of the AdS soliton spacetime is precisely 3/4 of the Casimir energy of the zero coupling gauge theory. The discrepancy of a factor 3/4 reflects the fact that two results apply in different regime of the dual gauge theory.

The above agreement is in favor of AdS/CFT correspondence in a non-supersymmetric case. But, the important question is whether the AdS soliton solution (2.1) is the lowest energy stable solution of the Einstein equations for the given boundary topology. On the basis of the stability of the non-supersymmetric field theory on $\mathbf{S^1 \times Y^2}$ together with the help of AdS/CFT correspondence, Horowitz and Myers have conjectured that the AdS soliton solution (2.1) is the minimum energy solution of the Einstein equations with that boundary condition. Any other solution with this boundary condition has positive energy with respect to (2.4). They have also shown that the candidate minimum energy solution is stable against all quadratic fluctuations of the metric.

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[4]The general case, *i.e.*, when the boundary topology is $R \times T^3$ or $R^2 \times T^2$ rather than $R^3 \times S^1$ has been studied in [18].

2.2 Phase Transition of Ricci-flat Black Hole

We shall now review how Ricci-flat black holes undergo a phase transition into the AdS soliton spacetime [12]. The Ricci-flat black hole metric is given by,

$$ds^2 = -\frac{r^2}{b^2}\left(1 - \frac{r_0^4}{r^4}\right)dt^2 + \frac{b^2}{r^2}\left(1 - \frac{r_0^4}{r^4}\right)^{-1}dr^2 + \frac{r^2}{b^2}d\theta^2 + \frac{r^2}{b^2}h_{ij}dx^i dx^j, \tag{2.9}$$

where r_0 is the position of the horizon and the periodic coordinate θ has a period η. Both the black hole metric and the AdS soliton metric are obtained varying the action,

$$I = -\frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left(R + \frac{12}{b^2}\right). \tag{2.10}$$

To study the thermodynamics of this black hole spacetime one has to go to the Euclidean theory by a Wick rotation $t \to i\tau$. The Euclidean metric is given by,

$$ds^2 = \frac{r^2}{b^2}\left(1 - \frac{r_0^4}{r^4}\right)d\tau^2 + \frac{b^2}{r^2}\left(1 - \frac{r_0^4}{r^4}\right)^{-1}dr^2 + \frac{r^2}{b^2}d\theta^2 + \frac{r^2}{b^2}h_{ij}dx^i dx^j \tag{2.11}$$

for the black hole spacetime and

$$ds_S^2 = \frac{r^2}{b^2}d\tau_S^2 + \frac{b^2}{r^2}\left(1 - \frac{\mu^4}{r^4}\right)^{-1}dr^2 + \frac{r^2}{b^2}\left(1 - \frac{\mu^4}{r^4}\right)d\theta_S^2 + \frac{r^2}{b^2}h_{ij}dx^i dx^j \tag{2.12}$$

for the soliton spacetime. The regularity of black hole spacetime at $r = r_0$ demands that the Euclidean time coordinate τ of the black hole is identified with period,

$$\beta = \frac{\pi b^2}{r_0}, \tag{2.13}$$

where the Euclidean time coordinate τ_S of the soliton space time has an arbitrary period β_S. It is worthwhile to mention here that $1/\beta$ and $1/\beta_S$ are the temperature of the black hole and soliton spacetime respectively.

We will now discuss the phase transition between the black hole spacetime and the AdS soliton spacetime. Given the boundary geometry $\mathbf{R} \times \mathbf{S^1} \times \mathbf{R^2}$ there are three solutions to the Einstein equations derived from the action (2.10). **(a)** Black hole spacetime (2.9), **(b)** soliton spacetime (2.1) and **(c)** global AdS spacetime, whose metric is given by,

$$ds_{AdS}^2 = -\frac{r^2}{b^2}dt_{AdS}^2 + \frac{b^2}{r^2}dr^2 + \frac{r^2}{b^2}d\theta_{AdS}^2 + \frac{r^2}{b^2}h_{ij}dx^i dx^j. \tag{2.14}$$

If we compute the on-shell action or free energy for black hole spacetime with respect to global AdS spacetime we find no signature of phase transition, *i.e.*, the black hole spacetime

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is always dominant over the global AdS spacetime. Similarly there exists no phase transition between the AdS soliton and the global AdS. The black hole free energy is given by,

$$F_B = -\frac{\pi^3 b^6 V}{16 G_5} \frac{\eta}{\beta^3} \qquad (2.15)$$

and the soliton free energy is given by,

$$F_S = -\frac{\pi^3 b^6 V}{16 G_5} \frac{\beta_S}{\eta_S^3}, \qquad (2.16)$$

where, we have used the global AdS as a reference point in both cases. Boundary topology must be the same for the black hole spacetime, the soliton spacetime and the global AdS spacetime, in order to compare them. Now if we compare the free energy of black hole spacetime and soliton spacetime then we can see that depending on the relative size of boundary $\mathbf{S^1}$ circles, the bulk spacetime is dominated by either black hole phase or the AdS soliton phase. So, in order to find possible phase transition between the black hole and the AdS soliton we have to compute the difference between black hole on-shell action (or free energy) and soliton on-shell action (or free energy).

The next step is to compute the difference between the on-shell black hole action and on-shell soliton action with the condition that the asymptotic boundary geometry of the black hole spacetime is same with that of soliton spacetime. This identification implies that,

$$
\begin{aligned}
\beta \sqrt{g_{\tau_B \tau_B}(\tilde{R})} &= \beta_S \sqrt{g_{\tau_S \tau_S}(\tilde{R})}, \\
\eta_S \sqrt{g_{\theta_S \theta_S}(\tilde{R})} &= \eta \sqrt{g_{\theta_B \theta_B}(\tilde{R})},
\end{aligned} \qquad (2.17)
$$

where, $\tilde{R}(\to \infty)$ is the position of the boundary hypersurface. Another important thing we should mention here is that integration over the radial coordinate r ranges from r_0 to $\tilde{R}$ for black hole spacetime and μ to $\tilde{R}$ for soliton spacetime. We will eventually take limit $\tilde{R} \to \infty$ at the end of our calculations. We can now calculate the regularized or subtracted action, which is given by,

$$
\begin{aligned}
I &= \left[I_B(\tilde{R}) - I_S(\tilde{R}) \right]_{\tilde{R} \to \infty} \\
&= \frac{\beta \, \eta_S \, V}{16 \pi G_5} \left(\frac{\mu^4}{b^2} - \frac{r_0^4}{b^2} \right),
\end{aligned} \qquad (2.18)
$$

where

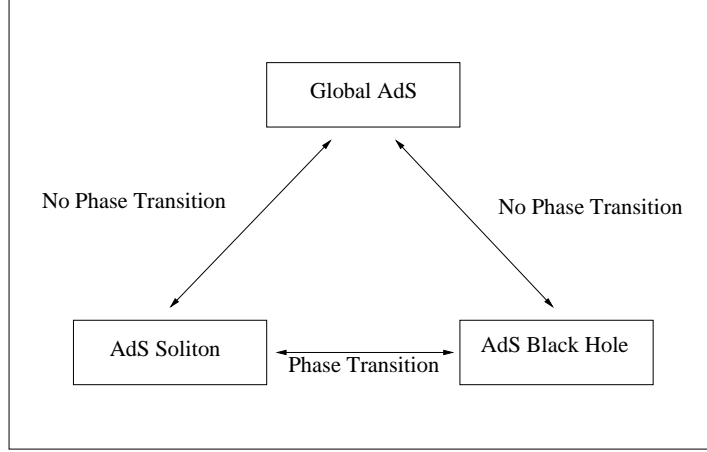$$V = \int \sqrt{h} dx^1 dx^2. \qquad (2.19)$$

Figure 1: Black Hole - Soliton Phase transition

Hence the free energy is,

$$
\begin{aligned}
F &= \beta^{-1} I \\
&= \frac{\eta_S \, V}{16 \pi G_5} \left(\frac{\mu^4}{b^2} - \frac{r_0^4}{b^2} \right) \\
&= \frac{\eta_S \, V}{16 \pi G_5} \frac{r_0^4}{b^2} \left(\frac{\mu^4}{r_0^4} - 1 \right).
\end{aligned}
\tag{2.20}
$$

We see that free energy is negative for $r_0 > \mu$ and positive for $r_0 < \mu$, signaling a phase transition. One can also express the free energy in terms of the temperature and the horizon area (size) of the black hole. It is easy to find the ratio,

$$
\frac{\mathcal{A}_B}{T^2} \sim \frac{r_0}{\mu},
\tag{2.21}
$$

where $\mathcal{A}_B$ is the area of the black hole horizon. It is obvious from equations 2.20 and (2.21) that the phase transition depends not only on the temperature of the black holes but also on the size of the black holes unlike the usual HP transition for spherical black holes where it completely depends on the temperature. The result can be summarized as follows. There are three phases in the bulk spacetime corresponding to a given boundary topology: the black hole phase, the global AdS phase and the AdS soliton phase. First and third phase are always dominant over the second one. But when boundary Euclidean time circle is less than the size of spatial $\mathbf{S^1}$ the black hole phase dominates in bulk otherwise the AdS soliton phase dominates, see figure 1.

Although discussing the phase transition was the main goal of this section, for completeness we will also write the expressions for thermodynamic variables, energy and entropy.

Using the following thermodynamic relations,

$$\begin{aligned} E &= \frac{\partial I}{\partial \beta}, \\ S &= \beta E - I, \end{aligned}$$

(2.22)

we find the energy and entropy as follows,

$$\begin{aligned} E &= \frac{V \eta_S}{16 \pi G_5 b^2} \left(3 r_0^4 + \mu^4 \right), \\ S &= \frac{\mathcal{A}_B}{4 G_5}. \end{aligned}$$

(2.23)

It is clear that energy is always positive with respective to the soliton.

We will conclude this section by mentioning the correspondence between this phase transition (in the bulk) and the phase transition of dual gauge theory on the boundary. For complete discussion the reader is referred to the original paper [12]. The stable black hole phase $r_0 > \mu$ corresponds to de-confined phase of the thermal gauge theory on $\mathbf{S^1} \times \mathbf{S^1} \times \mathbf{R^2}$ and the stable soliton phase $r_0 < \mu$ corresponds to confined phase of the thermal gauge theory on $\mathbf{S^1} \times \mathbf{S^1} \times \mathbf{R^2}$. It is important to mention here that only the antiperiodic spin structure $(Tre^{-\beta H})$ in both the $\mathbf{S^1}$ directions undergoes the large N phase transition.

3   Phase Transition of Charged Black Holes

So far we have seen that neutral Ricci-flat AdS black holes undergo a phase transition into a global soliton spacetime. Therefore, it would be interesting to generalize this idea to the case of charged black holes.

Electrically charged black holes in five dimensions have drawn a lot of interest in the context of AdS/CFT. The electric charge of these black holes are mapped to the global R-charge of the dual field theory. Because of the presence of the electric charges, the thermodynamics and the phase structure of these black holes are rather complicated and also interesting at the same time. There have been a lot of study of thermodynamics and phase transitions of these charged black hole with different horizon topologies (see [19, 20, 21] and references therein). But it seems that the study of thermodynamics and phase transition between charged Ricci-flat black holes and AdS soliton spacetime are yet to be explored. In this section we will shed some light on these issues.

We will first briefly discuss how charge black holes can arise in five dimensions in the context of string theory. A consistent truncation of $\mathcal{N} = 8$, $D = 5$ gauged supergravity

with $SO(6)$ Yang-Mills gauge group, which can be obtained by S^5 reduction of type IIB supergravity, gives rise to $\mathcal{N} = 2$, $D = 5$ gauge supergravity with $U(1)^3$ gauge group. The same theory can also be obtained by compactifying eleven dimensional supergravity, low energy theory of M theory, on a Calabi-Yau three folds. The bosonic part of the action of $\mathcal{N} = 2$, $D = 5$ gauged supergravity is given by,

$$
\begin{aligned}
I_{sugra} &= \int d^5x \sqrt{-g} \left(\frac{R}{16\pi G_5} + \frac{V(X)}{b^2} - \frac{1}{2} G_{IJ}(X) \partial_\mu X^I \partial^\mu X^J - \frac{1}{4} G_{IJ}(X) F_{\mu\nu}^I F^{\mu\nu\ J} \right) \\
&+ \ C.S. \ terms,
\end{aligned}
\tag{3.1}
$$

where, X^I's are three real scalar fields, subject to the constraint $X^1 X^2 X^3 = 1$. F^I's are field strengths of three Abelian gauge fields (I,J=1,2,3). The scalar potential $V(X)$ is given by,

$$
V(X) = 2 \left(\frac{1}{X^1} + \frac{1}{X^2} + \frac{1}{X^3} \right).
\tag{3.2}
$$

The metric on the scalar manifold G_{IJ} is given by,

$$
G_{IJ}(X) = \frac{1}{2} diag \left(\frac{1}{(X^1)^2}, \frac{1}{(X^2)^2}, \frac{1}{(X^3)^2} \right).
\tag{3.3}
$$

The solution of this action is specified by the following metric,

$$
ds^2_{sugra} = -(H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3} \frac{dr^2}{f} + (H_1 H_2 H_3)^{1/3} \frac{r^2}{b^2} d\Omega_k^2,
\tag{3.4}
$$

where

$$
f = k - \frac{M}{r^2} + \frac{r^2}{b^2} H_1 H_2 H_3, \quad H_I = 1 + \frac{\tilde{q}_I}{kr^2},
\tag{3.5}
$$

The three real scalar fields X^I's and gauge potentials A_μ^I's are of the form,

$$
X^I = H_I^{-1} (H_1 H_2 H_3)^{1/3}, \quad A_t^I = -\frac{\sqrt{k\tilde{q}_I(\tilde{q}_I + M)}}{kr^2 + \tilde{q}_I} + \Phi^I,
\tag{3.6}
$$

where Φ^I's are constants. $k = 1, 0, -1$ corresponds to black holes with spherical, Ricci-flat and hyperboloid horizon topology respectively. We are interested in k=0 case. This is done by taking the limit $k \to 0$, $\tilde{q}_I \to 0$ with $q_I = \tilde{q}_I/k$ fixed. Then the solution becomes,

$$
\begin{aligned}
ds^2_{sugra,k=0} &= -(H_1 H_2 H_3)^{-2/3} f dt^2 + (H_1 H_2 H_3)^{1/3} \frac{dr^2}{f} \ + \ (H_1 H_2 H_3)^{1/3} \frac{r^2}{b^2} d\theta^2 \\
&\qquad\qquad\qquad\qquad\qquad + \ (H_1 H_2 H_3)^{1/3} \frac{r^2}{b^2} d\vec{x}^2,
\end{aligned}
$$

$$
f = -\frac{M}{r^2} + \tilde{g}^2 r^2 H_1 H_2 H_3, \quad X^I = H_I^{-1}(H_1 H_2 H_3)^{1/3}, \quad A_t^I = -\frac{\sqrt{q_I M}}{r^2 + q_I} + \Phi^I, \tag{3.7}
$$

where θ is an angular coordinate with period η. We will choose Φ^I's in such a way that the gauge potential vanishes on the horizon r_0,

$$\Phi^I = \frac{\sqrt{q_I M}}{r_0^2 + q_I},\tag{3.8}$$

where horizon radius r_0 is given by the solution of the following equation,

$$f(r_0) = 0.\tag{3.9}$$

It turns out that asymptotic values Φ_I's of gauge fields A_I's behave as chemical potential when we consider the black hole thermodynamics in grand canonical ensemble.

At this point we would like to add a remark that $k = 0$ three charge AdS black holes of $\mathcal{N} = 2$ gauged supergravity in $D = 5$ can be embedded in $D = 10$ as a solution that is precisely the decoupling limit of the rotating $D3$ brane [20, 22].

One special case, namely three equal charge case ($q_1 = q_2 = q_3 = q$), of this black hole solution is actually the same as the AdS-Reissner-Nordström black hole solution described in equations (3.11)-(3.13). In this case scalar fields become constant and the scalar potential in the action reduces to a constant value. The action becomes the Einstein-Maxwell action in the AdS spacetime. Though the metric looks somewhat different than usual AdS-Reissner-Nordström black hole metric, after a suitable coordinate transformation one can write this metric in usual Schwarzschild coordinate. It is important to mention here that although AdS-Reissner-Nordström black holes and three equal charge R-charged back holes show the same kind of phase transition, their local stability behavior is somewhat different from each other. We will discuss these issues later in this paper.

In the next subsection we discuss phase transition and thermodynamics of Ricci-flat AdS-Reissner-Nordström black hole, which is somewhat simpler than general R-charge black hole, and will be a warm-up example. In subsection 3.2 we will focus on general R-charge black holes.

3.1   AdS-Reissner-Nordström black holes

AdS-Reissner-Nordström black holes are the solution of field equations governed by the Einstein-Maxwell action with negative cosmological constant. The action is given by,

$$I = -\frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left(R + \frac{12}{b^2} - F^2 \right).\tag{3.10}$$

The solution is specified by the a metric and a gauge field. The metric is given by,

$$ds^2 = -V(r)dt^2 + \frac{dr^2}{V(r)} + \frac{r^2}{b^2}d\theta^2 + \frac{r^2}{b^2}(dx_1^2 + dx_2^2), \tag{3.11}$$

where $V(r)$ is given by,

$$V(r) = -\frac{m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{b^2}, \tag{3.12}$$

and m and q are two constants of integration (parameters). Later we will relate them to the ADM mass and physical charge of the black hole respectively. The gauge field is given by,

$$A(r) = \left(-\frac{1}{c}\frac{q}{r^2} + \Phi\right) dt, \tag{3.13}$$

where $c = 2/\sqrt{3}$ and Φ is constant. We will choose Φ in such a way that $A(r_0) = 0$ and this gives

$$\Phi = \frac{1}{c}\frac{q}{r_0^2}. \tag{3.14}$$

where r_0 is the position of horizon and is given by the solution of the following equation,

$$V(r_0) = 0.$$

The coordinate θ is periodic and has a period η. This five dimensional spacetime (X) has a conformal boundary $M = R \times S^1 \times R^2$.

In order to consider thermodynamics of this black hole in the classical limit we will work with the Euclidean theory which is obtained by a Wick rotation $t \to i\tau$. The Euclidean metric is given by,

$$ds^2 = V(r)d\tau^2 + \frac{dr^2}{V(r)} + \frac{r^2}{b^2}d\theta^2 + \frac{r^2}{b^2}(dx_1^2 + dx_2^2) \tag{3.15}$$

and the gauge field becomes,

$$A_\tau = i\left(-\frac{q}{cr^2} + \frac{q}{cr_0^2}\right) \tag{3.16}$$

The Hawking temperature of the black hole is given by,

$$T = \frac{2r_0^6 - q^2b^2}{2\pi b^2 r_0^5} \tag{3.17}$$

We will now calculate the on-shell black hole action, (3.10), which is given by,

$$I_B = \frac{\beta\,\eta\,V}{16\pi G_5 b^3}\left(\frac{2(\tilde{R}^4 - r_0^4)}{b^2} - \frac{2q^2}{r_0^2}\right), \tag{3.18}$$

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where $\tilde{R}$ is the cutoff in the radial direction and $V = \int dx_1 dx_2$ (2.19). As $\tilde{R} \to \infty$, the on-shell action equation (3.18) diverges. To study the phase transition between the black hole and AdS soliton we will subtract the contribution of the AdS-soliton from black hole action and it will also regularize the onshell action.

As we mentioned earlier given the boundary topology to be $R \times S^1 \times R^2$ (Lorentzian), the AdS-Soliton is conjectured to be the minimum energy solution of the action (3.10). The solution is given by a constant gauge field and a metric (2.1),

$$ds^2_S = -dt^2_S + \frac{dr^2}{V_S(r)} + V_S(r)d\theta^2_S + \frac{r^2}{b^2}(dx^2_1 + dx^2_2) \tag{3.19}$$

where

$$V_S(r) = \frac{r^2}{b^2}\left(1 - \frac{\mu^4}{r^4}\right). \tag{3.20}$$

The coordinate θ_S has period,

$$\eta_S = \frac{\pi b^2}{\mu}. \tag{3.21}$$

The Euclidean soliton metric is given by (2.12),

$$ds^2_S = \frac{r^2}{b^2}d\tau^2_S + \frac{dr^2}{V_S(r)} + V_S(r)d\theta^2_S + \frac{r^2}{b^2}(dx^2_1 + dx^2_2), \tag{3.22}$$

where τ_S can have an arbitrary period β_S. Now we can calculate the on shell AdS-Soliton action which is given by,

$$I_S = \frac{\beta_S \, \eta_S \, V}{16\pi G_5 b^3}\left(\frac{2(\tilde{R}^4 - \mu^4)}{b^2}\right). \tag{3.23}$$

Before subtracting the AdS-Soliton contribution from the black hole contribution we should remember that in order to match the boundary geometry of black hole spacetime to that of the soliton spacetime we have the relations (2.17) [5], which in the present case reduces to

$$\beta\sqrt{V(\tilde{R})} = \frac{\tilde{R}}{b}\beta_S,$$
$$\beta_S = \beta\left(1 - \frac{m \, b^2}{2\tilde{R}^4}\right) \tag{3.24}$$

and

$$\eta_S\sqrt{V_S(\tilde{R})} = \frac{\tilde{R}}{b}\eta,$$
$$\eta = \eta_S\left(1 - \frac{\mu^4}{2\tilde{R}^4}\right). \tag{3.25}$$

---

[5]In addition to these relations we also have to identify the chemical potential of black hole spacetime to that of soliton spacetime. Since AdS soliton is a solution of the Einstein equations with constant gauge potential, we can assign any chemical potential for this spacetime.

Using these relations we can now calculate the regularized or subtracted action,

$$
\begin{aligned}
I & = \left[I(\tilde{R}) - I(\tilde{R}) \right]_{\tilde{R} \to \infty} \\
& = \frac{\beta\, \eta_S\, V}{16\pi G_5} \left(\frac{\mu^4}{b^2} - \frac{2q^2}{r_0^2} - \frac{2r_0^4}{b^2} + m \right) \\
& = \frac{\beta\, \eta_S\, V}{16\pi G_5} \left(\frac{\mu^4}{b^2} - \frac{q^2}{r_0^2} - \frac{r_0^4}{b^2} \right).
\end{aligned}
\tag{3.26}
$$

And hence the Gibbs free energy is given by,

$$
\begin{aligned}
G & = TI \\
& = \frac{\eta_S\, V}{16\pi G_5} \left(\frac{\mu^4}{b^2} - \frac{q^2}{r_0^2} - \frac{r_0^4}{b^2} \right).
\end{aligned}
\tag{3.27}
$$

3.1.1 Thermodynamics

In this subsection we will discuss the thermodynamics of AdS-Reissner-Nordström black holes. There are two kinds of ensembles one can use to describe the thermodynamics of a charged black hole system. (i) Canonical ensemble - where temperature of the system is fixed but energy can flow between the system and the heat bath. (ii) Grand canonical ensemble - where temperature and the electric potential are fixed and energy and charge can flow.

In this paper we will study thermodynamics considering the system in a grand canonical ensemble, since it is interesting in the context of AdS/CFT.

A state in grand-canonical ensemble is characterized by (inverse) temperature β and electric potential Φ. In the grand-canonical ensemble the Gibbs potential is given by,

$$
G = \frac{I}{\beta} = E - TS - \Phi Q,
\tag{3.28}
$$

where E is the energy, S is the entropy and Q is physical charge of the system. Using the above relation we may compute the variables of the system as follows,

$$
\begin{aligned}
E & = \left(\frac{\partial I}{\partial \beta} \right)_\Phi - \frac{\Phi}{\beta} \left(\frac{\partial I}{\partial \Phi} \right)_\beta, \\
S & = \beta \left(\frac{\partial I}{\partial \beta} \right)_\Phi - I, \\
Q & = -\frac{1}{\beta} \left(\frac{\partial I}{\partial \Phi} \right)_\beta.
\end{aligned}
\tag{3.29}
$$

Together, they satisfy the first law,
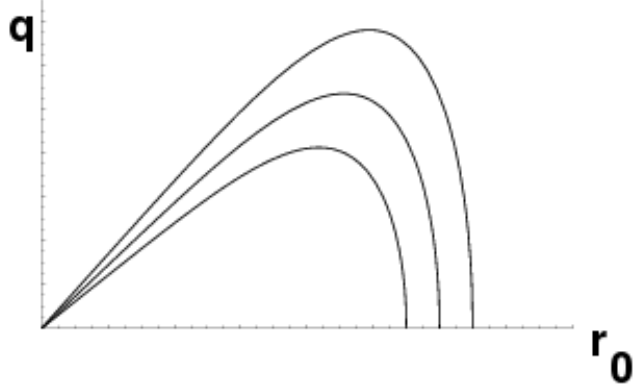
$$
dE = TdS + d(\Phi Q).
\tag{3.30}
$$

Figure 2: Phase transition curves (equation 3.34) in $q - r_0$ plane for AdS-Reissner-Nordström black hole. Three different curves for three different values of μ. As we increase the value of μ corresponding phase transition curves go away from the r_0 axis.

Using the Euclidean action (equation 3.26) and the above relations we compute the thermodynamic variables as follows,

$$E = \frac{V\eta_S}{16\pi G_5}\left(3m + \frac{\mu^4}{b^2}\right), \tag{3.31}$$

$$S = \frac{1}{4G_5}\frac{V\eta_S r_0^3}{b^3} = \frac{A}{4G_5}, \tag{3.32}$$

$$Q_{phy} = \frac{\sqrt{3}}{4\pi G_5}\frac{V\eta_S}{b^3}q. \tag{3.33}$$

Equation (3.31) shows that the energy is always positive with respect to the soliton energy.

3.1.2   Phase Transition

Clearly the Gibbs free energy (3.27) carries the signature of phase transition. When $G < 0$ the black hole phase dominates. On the other hand, when $G > 0$, the black hole phase is unstable and decays to the soliton. Once we fix the size of the boundary spatial circle, *i.e.*, fixed η (and so fixed η_S or μ), the phase transition curve depends on the charge 'q' and the size of the black hole 'r_0'. Conditions are given by the following relations,

$$r_0^6 - \mu^4 r_0^2 + q^2 b^2 > 0 \qquad black \; hole \; phase \; dominates,$$
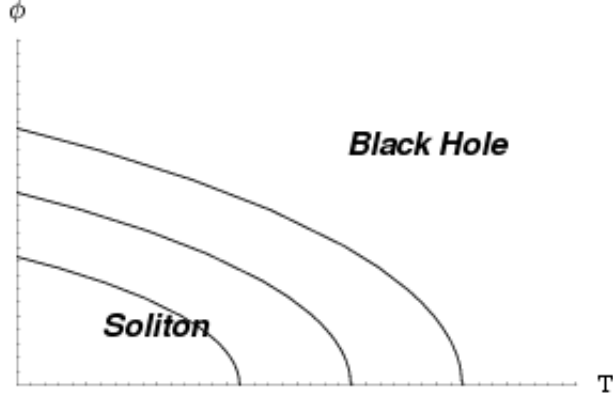$$r_0^6 - \mu^4 r_0^2 + q^2 b^2 < 0 \qquad AdS \; soliton \; phase \; dominates. \tag{3.34}$$

Figure 3: T_{tran} Vs Φ Plot for AdS-Reissner-Nordström black hole. Different curves correspond to different μ. For smaller value of μ the corresponding phase transition curves are closer to origin. When $T > T_{tran}$ BH spacetime has dominant contribution to Euclidean path integral

Since a system in grand-canonical ensemble is specified by its temperature and chemical potential, it would be interesting to plot the phase transition diagram in $T - \Phi$ plane. For a given chemical potential Φ , the temperature can be written as,

$$T = \frac{3r_0^2 - 2b^2\Phi^2}{3\pi b^2 r_0}. \tag{3.35}$$

Using the relation 3.34, we can write the relation between transition temperature T_{tran}, chemical potential Φ and μ,

$$
\begin{aligned}
T_{tran} &= \sqrt{\frac{-4b^2\Phi^2 + \sqrt{16b^4\Phi^4 + 36\mu^4}}{6}} \frac{1}{\pi b^2} \\
&\quad - \frac{2\Phi^2}{3\pi}\left(\frac{-4b^2\Phi^2 + \sqrt{16b^4\Phi^4 + 36\mu^4}}{6}\right)^{-1/2}.
\end{aligned} \tag{3.36}
$$

Fig. 3 depicts how transition temperature changes with chemical potential Φ for different μ.

We will end this subsection briefly mentioning the effect of Gauss-Bonnet term to the action 3.10. The solution is give in [23]. Since Gauss-Bonnet term does not change the thermodynamics (energy, entropy and physical charge) for Ricc-flat black holes we are not presenting the details of our result here. Phase transition condition also does not receive any correction in presence of the Gauss-Bonnet term in the action. One can check that the functional dependence of free energy (which is proportional to difference between on shell

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black hole action and on shell soliton action) on charge q horizon radius r_0 and radius of boundary spatial circle is same as equation 3.27, only the over all coefficient changes.

3.2 R-charge Black Holes

In this subsection we will focus on phase transition of $D = 5$ R-charge black holes. First we will give a general expression of free energy for any arbitrary three charge case. In order to study the phase transition and stability we will focus on three different limits, (i) one charge case, $i.e.$, $q_1 = q, q_2 = q_3 = 0$, (ii) two equal charge case, $q_1 = q_2 = q, q_3 = 0$ and (iii) three equal charge case, $q_1 = q_2 = q_3 = q$.

In order to calculate free energy for these black holes we will follow same steps as in the last subsection. But here we will encounter a subtlety. In the previous case both the black hole and the soliton metric were written in Schwarzschild coordinate and hence we had calculated both the on-shell action and the background action keeping in mind relations (3.24) and (3.25). But in this case the black hole metric (3.4) is written in the "isotropic coordinates" but the soliton metric (2.1) is written in the usual Schwarzschildcoordinates. So either we have to write the black hole metric in the Schwarzschild coordinate by suitably changing the coordinates and redefining the parameters or we can write the soliton metric in the isotropic coordinate by redefining the coordinate as follows,

$$r^2 \rightarrow r^2(H_1 H_2 H_3)^{\frac{1}{3}}. \tag{3.37}$$

We shall follow the second approach. After this coordinate change the soliton metric becomes,

$$
\begin{aligned}
ds_S^2 &= -\frac{r^2}{b^2}(H_1 H_2 H_3)^{1/3} dt_S^2 + \frac{b^2}{r^2}\left(1 - \frac{\frac{q1}{H_1} + \frac{q2}{H_2} + \frac{q3}{H_3}}{3r^2}\right)^2 \left(1 - \frac{\mu^4}{r^4(H_1 H_2 H_3)^{2/3}}\right)^{-1} dr^2 \\
&+ \frac{r^2}{b^2}(H_1 H_2 H_3)^{1/3}\left(1 - \frac{\mu^4}{r^4(H_1 H_2 H_3)^{2/3}}\right) d\Theta_S^2 + \frac{r^2}{b^2}(H_1 H_2 H_3)^{1/3} d\vec{x}^2.
\end{aligned}
\tag{3.38}
$$

The periodicity η_S of the compact direction θ_S remains same as (2.3). We will now compute the on-shell black hole action (3.1) and subtract contribution of the global soliton action from this to get finite or renormalized on-shell action. In this case, unlike the AdS-Reissner-Nordström spacetime, the Gibbons-Hawking boundary (GH) terms also give a finite contribution to the on-shell action.

To study thermodynamics and phase transition we have to go to the Euclidean space by

Wick rotation $t \to i\tau$. The inverse Hawking temperature of the black hole is given by,

$$\beta = \frac{2\pi b^2 r_0^2 \sqrt{(r_0^2 + q_1)(r_0^2 + q_2)(r_0^2 + q_3)}}{2r_0^6 + r_0^4(q_1 + q_2 + q_3) - q_1 q_2 q_3}. \tag{3.39}$$

Detailed derivation of the on-shell action has been given in appendix A. Here we will only write the result (from here we will work in $b = 1$ unit for simplicity).

$$I = -\frac{\beta V \eta_S}{16\pi G_5} \left(M + \frac{2}{3}((q_1^2 + q_2^2 + q_3^2) - (q_1 q_2 + q_2 q_3 + q_3 q_1)) - \mu^4 \right). \tag{3.40}$$

Using this regularized action if we calculate mass of these black holes, an unexpected nonlinear term involving charges appears [24]. There is nothing inherently wrong in the nonlinear appearance of charge in the expression of mass but it can contradict the expected BPS inequality between the charge and the mass. Liu and Sabra [25] pointed out that inclusion of a finite counterterm can resolve that problem. The counterterm they proposed is,

$$I_C = \frac{1}{8\pi G_5} \int d^4 x \sqrt{-h} \vec{\phi}^2, \tag{3.41}$$

where $\vec{\phi}$ is related to three scalar fields in the following way,

$$X_I = e^{-\frac{1}{2} \vec{a}_I \cdot \vec{\phi}},$$

with the condition, $\vec{a}_I \cdot \vec{a}_J = 4\delta_{IJ} - 4/3$ [22]. It is straightforward to check that using this field redefinition, the counterterm becomes,

$$I_C = \frac{\beta V \eta_S}{16\pi G_5} \left(\frac{2}{3}((q_1^2 + q_2^2 + q_3^2) - (q_1 q_2 + q_2 q_3 + q_3 q_1)) \right). \tag{3.42}$$

After adding this counterterm we can get rid of that nonlinear charge term in the action. So the final action is given by,

$$I = -\frac{\beta V \eta_S}{16\pi G_5} \left(M - \mu^4 \right) \tag{3.43}$$

and the Gibbs free energy is given by,

$$\begin{aligned} G &= -\frac{V \eta_S}{16\pi G_5} \left(M - \mu^4 \right) \\ &= -\frac{V \eta_S}{16\pi G_5} \left(r_0^4 H_1(r_0) H_2(r_0) H_3(r_0) - \mu^4 \right). \end{aligned} \tag{3.44}$$
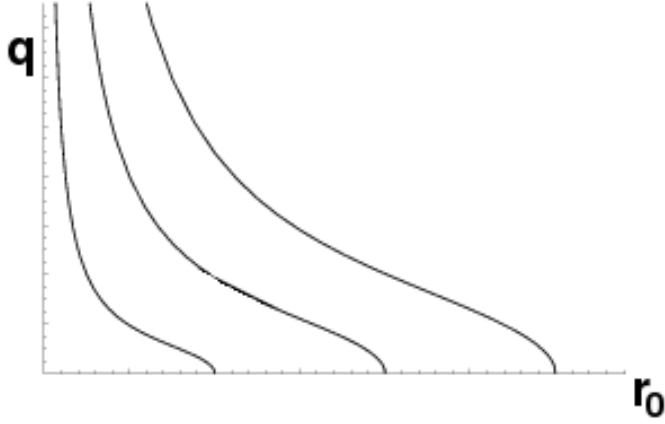
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Figure 4: Phase transition curves (equation 3.49) for different μ in $q - r_0$ plane for single charge black holes. As value of μ increases transition curves recede from the origin.

3.2.1 Thermodynamics

We will use the set of thermodynamic relations given by equation (3.29) and Gibbs potential (3.28) to compute energy, entropy and physical charge of R-charge black holes.

$$E = \frac{V\eta}{16\pi G_5} \left[3M + \mu^4 \right],$$

(3.45)

$$S = \frac{V\eta}{4G_5} \sqrt{(r_0^2 + q_1)(r_0^2 + q_2)(r_0^2 + q_3)} = \frac{Area}{4G_5},$$

(3.46)

$$Q^I_{phys} = \frac{V\eta}{8\pi G_5} \sqrt{q_I(r_0^2 + q_1)(r_0^2 + q_2)(r_0^2 + q_3)/r_0^2}.$$

(3.47)

3.2.2 Phase Transition

Signature of phase transition is obvious from the expression of Gibbs free energies (equation 3.43) and the phase transition depends on the size of compact dimension. The phase transition conditions are given by,

$$
\begin{aligned}
r_0^4 H_1(r_0) H_2(r_0) H_3(r_0) &> \mu^4 \quad Black\ Hole\ Phase\ dominates, \\
r_0^4 H_1(r_0) H_2(r_0) H_3(r_0) &< \mu^4 \quad Soliton\ Phase\ dominates.
\end{aligned}
$$

(3.48)

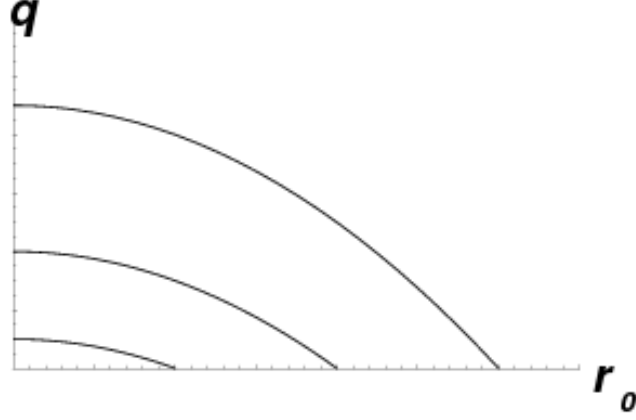We will now discuss the phase diagram for this black hole and concentrate on three special cases.

Figure 5: Phase transition curves (equation 3.50) for different μ in $q - r_0$ plane for two equal charge black holes.

A. $q_1 = q$, $q_2 = q_3 = 0$

In this case the phase transition conditions are given by,

$$
\begin{aligned}
r_0^4 + q r_0^2 - \mu^4 &> 0 \quad Black \ \ Hole \ \ Phase \ \ dominates, \\
r_0^4 + q r_0^2 - \mu^4 &< 0 \quad AdS \ \ Soliton \ \ Phase \ \ dominates.
\end{aligned}
\tag{3.49}
$$

Figure 4 shows the phase transition curves for different values of μ in $q - r_0$ plane.

B. $q_1 = q$, $q_2 = q$ and $q_3 = 0$

For two equal charges phase transition condition is given by equation,

$$
\begin{aligned}
q &> \mu^2 - r_0^2 \quad Black \ \ Hole \ \ Phase \ \ dominates, \\
q &< \mu^2 - r_0^2 \quad AdS \ \ Soliton \ \ Phase \ \ dominates.
\end{aligned}
\tag{3.50}
$$

Phase transition curves have been plotted in figure 5.

C. $q_1 = q$, $q_2 = q$ and $q_3 = q$

And finally for three equal charge black holes phase transition condition is given by,

$$
\begin{aligned}
q &> \mu^{4/3} r_0^{2/3} - r_0^2 \quad Black \ \ Hole \ \ Phase \ \ dominates, \\
q &< \mu^{4/3} r_0^{2/3} - r_0^2 \quad AdS \ \ Soliton \ \ Phase \ \ dominates.
\end{aligned}
\tag{3.51}
$$
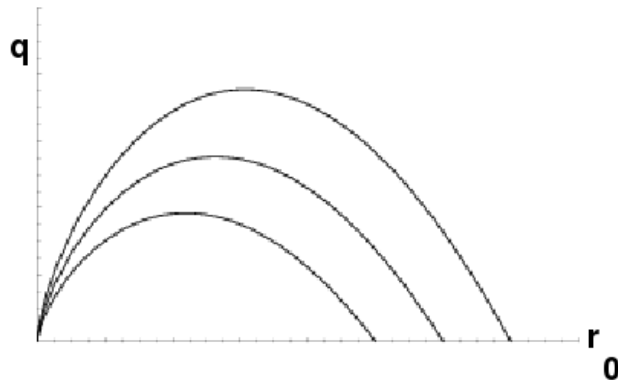
Phase transition curves have been plotted in figure 6.

Figure 6: Phase transition curves (equation 3.51) for different μ in $q - r_0$ plane for three equal charge charge black holes.

4 Local Stability of Charged Ricci-flat Black Holes

Finally we discuss local thermodynamic stability of charged black holes we were considering in the previous section. Local thermodynamic stability of a system implies that the entropy S which is a function of other extensive thermodynamic variables x_i's of the system is subadditive in a sufficiently small neighborhood of a given point in the phase space of x_i's. The criterion of subadditivity is,

$$S(\lambda x_i + (1 - \lambda)x_i) \geq \lambda S(x_i) + (1 - \lambda)S(x_i), \quad 0 \leq \lambda \leq 1. \tag{4.1}$$

If the inequality goes the other way, then the system can gain entropy by dividing into two parts, one with a fraction λ of the energy, charge, etc and the other with a fraction $1 - \lambda$. But no such process is allowed by second law of thermodynamics, and hence the system is thermodynamically stable [19, 26]. When S is a smooth function of x_i's then sub-additivity is equivalent to the Hessian matrix $\left[\frac{\partial^2 S}{\partial x_i \partial x_j}\right]$ being negative definite.

Now one has to decide among all the extensive quantities x_i's which are thermodynamic variables of the system, *i.e.*, they can vary in an experiment and which are the fixed parameters of the system. For example, in the canonical ensemble, mass (energy) M is the only thermodynamic variable, *i.e.*, the system can interchange energy with the heat bath but temperature and charges are the fixed quantities. A canonical system is specified by its temperature and charges. For grand canonical ensemble mass (energy) M and charges Q_I's are thermodynamic variables, *i.e.*, the system can interchange energy and charge with the

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heat bath and temperature and chemical potential are constant parameters. In this case, the phase space is specified by mass and charge.

So in the grand canonical ensemble the lines of instability in the phase space are determined by finding the zeros of the determinant of the Hessian sub-matrix $\left[\frac{\partial^2 S}{\partial x_i \partial x_j}\right]$, where, x_i's are mass and charges. For the AdS-Reissner-Nordström black holes, the Hessian is a 2×2 matrix and for R-charge black holes this is a 4×4 matrix. It has been argued in [26] that the zeros of the determinant of the Hessian of S with respect to M and Q_I's coincide with the zeros of the determinant of the Hessian of the Gibbs (Euclidean) action,

$$I_G = \beta \left(M - \sum_{I=1}^{3} \Phi_I Q_I \right) - S, \tag{4.2}$$

with respective to r_0 and q_I's keeping β and Φ_I's fixed. Note that q_I's are the charge parameters entering into the black hole solutions where Q_I's are the physical charges. Though this criteria can figure out the instability line in the phase diagram but it is unable to tell which sides of the phase transition lines correspond to local stability. One can figure out the stability region by knowing the fact that zero chemical potential and high temperature must correspond to a stable black hole solution.

Using the procedure stated in the last paragraph we will find out the region of stability for the black holes we have discussed. We will first consider stability of the AdS-Reissner-Nordström black holes and then focus on three special cases of R-charged black holes, namely one charge, two equal charged and three equal charged black holes, for simplicity. The general case can also be done using the expressions for mass, charge and entropy, given in equations (3.45, 3.46, 3.47).

A. AdS-Reissner-Nordström Black Holes

AdS-Reissner-Nordström black hole is a single charge black hole. The Hessian is a 2×2 matrix. Determinant of this matrix vanishes at $r_0 = 0$. So in $r_0 - q$ plane the line $r_0 = 0$ is the line of instability, and the black hole is stable at any $r_0 > 0$ and $q > 0$ point .

For AdS-Reissner-Nordström black holes q must be less than $2r_0^2$, otherwise the black hole will have negative temperature. $q = 2r_0^2$ corresponds to $T = 0$ line in $T - \Phi$ plane. So the black hole is stable at any non-zero temperature and chemical potential. No instability line is found in the $T - \Phi$ plane.
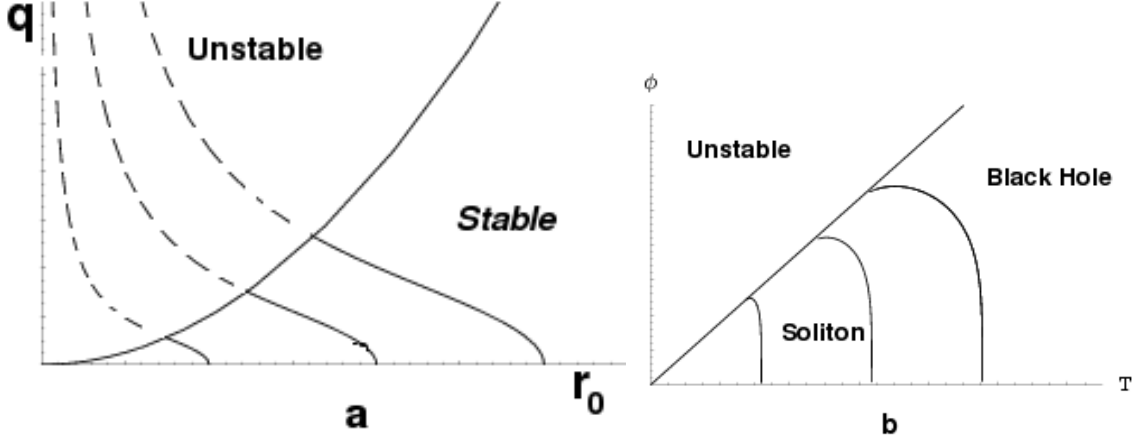
Figure 7: Fig. **(a):** Phase transition curves for different μ and instability line in $r_0 - q$ plane for single charge charge black holes. $q = 2r_0^2$ is the instability line. Phase transition lines starts for $q = 0$ and $r_0 = \mu$. As we move along the phase transition line r_0 decreases and q increases and when q becomes equal to $2r_0^2$ phase transition lines touch the instability lines. After that $q > 2r_0^2$, and phase transition curves enter into the unstable region (dashed lines). Fig.**(b):** Same curves have been plotted in $T - \Phi$ plane.

B. $\mathbf{q_1 = q, \ \ q_2 = q_3 = 0}$

In this case also the Hessian is a 2×2 matrix. Zeros of the determinant are given by the following condition,

$$q = 2r_0^2. \tag{4.3}$$

The region of stability is determined by the condition, $q < 2r_0^2$ where $q = 2r_0^2$ is the line of instability. We have plotted the instability lines and phase transition line in $q - r_0$ plane as well as in $T - \Phi$ plane in figure 7. In the $T - \Phi$ plane the instability line is a straight line and the region,

$$\frac{\Phi}{T} < \frac{\pi}{\sqrt{2}} \tag{4.4}$$

is stable [27].

C. $\mathbf{q_1 = q, \ \ q_2 = q, \ \ q_3 = 0}$

To find out the stability of two equal charge black holes, we have to keep it in mind that, first we will find out the Hessian matrix with two independent non zero charges and then set them equal. So in this case the Hessian is a 3×3 matrix. Zeros of its determinant are given by,
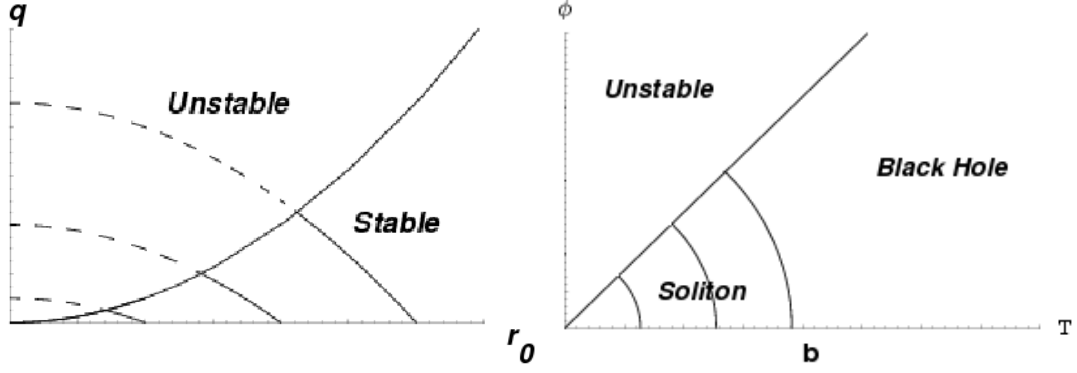
$$q = r_0^2. \tag{4.5}$$

Figure 8: Fig.(a):Phase transition curves for different μ and instability line in $r_0 - q$ plane for two equal charge charge black holes. $q = r_0^2$ is the instability line. Fig. (b): Phase transition curves and instability line $T - \Phi$ plane.
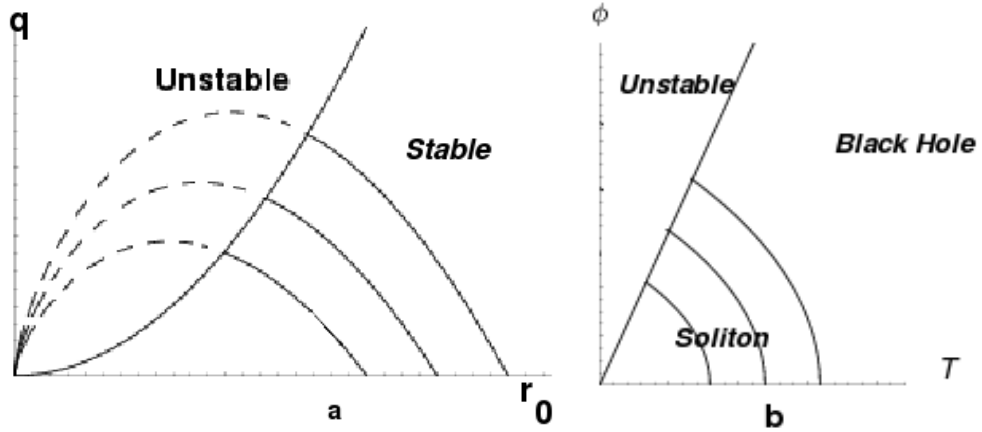

Figure 9: Phase transition and stability curves for three equal charge black holes. Fig.(a): Phase transition curves for different μ in $r_0 - q$ plane. Instability lines is $q = r_0^2$. Fig. (b): Phase transition curves and instability line $T - \Phi$ plane.

The region of stability is

$$q < r_0^2 \tag{4.6}$$

in $r_0 - q$ plane and

$$\frac{\Phi}{T} < \pi \tag{4.7}$$

in $T - \Phi$ plane. See figure 8.

D. $q_1 = q$, $q_2 = 0$, $q_3 = q$

Similarly for three equal charge case also we will first compute the Hessian for three independent charges, so the Hessian is a 4×4 matrix. Then we set three charges to be equal. The determinant vanishes for,

$$q = r_0^2. \tag{4.8}$$

Hence $q = r_0^2$ determines the instability line and

$$q < r_0^2 \tag{4.9}$$

is the region of stability in $r_0 - q$ plane. Similar plot exists in the $T - \Phi$ plane, where stability region is determined by [figure 9],

$$\frac{\Phi}{T} < 2\pi. \tag{4.10}$$

5 Discussion

We have investigated the nature of phase transition curves of charged Euclidean black holes in five dimensions whose asymptotic boundary topology is $\mathbf{S^1} \times \mathbf{S^1} \times \mathbf{R^2}$. Bulk spacetime with this boundary topology does not show any signature of phase transition between black hole and a global AdS spacetime. The Gibbs free energy remains always negative for any positive value of r_0 and q, and hence black hole phase is always dominant in the Euclidean path integral. But we have shown that instead of global AdS if we compare the black hole with the AdS soliton then the black hole could undergo a phase transition. Depending on the size of the boundary $\mathbf{S^1}$ circle the Gibbs potential flips sign as we vary r_0 and q or temperature (T) and chemical potential (Φ).

First we have considered the simplest five dimensional charged black holes which are AdS-Reissner-Nordström black holes and studied their thermodynamics and phase transition. We then focused on various five dimensional R-charged black holes which arise as a solution of

$\mathcal{N} = 2$ gauged supergravity in five dimensions. We have shown in all four cases that the phase transition lines in the phase diagrams typically depend on the size of the boundary S^1 circle. As a consistency check, if we set μ, which is proportional to inverse radius of boundary circle, to zero then the radius of $\mathbf{S^1}$ circle becomes infinity, the boundary space becomes $\mathbf{R^3}$ and all the phase transition lines disappear from the phase diagram, as expected.

We have also discussed the stability of these Ricci-flat charged black holes and shown that the R-charged black holes are locally stable in some region of the phase space. The black hole instability line rises linearly with temperature in $T - \Phi$ plane and the slope is $\frac{\pi}{\sqrt{2}}$, π and 2π for single charge black holes, two equal charge black holes and three equal charge black holes respectively. The instability lines do not depend on the size of the boundary $\mathbf{S^1}$ circle. It depends on topology of boundary spacetime, *i.e.*, whether the the boundary spacetime is flat ($k = 0$), spherical ($k = 1$) or hyperboloid ($k = -1$). For spherical black holes instability lines are found in [19]. In [19] instability lines were also found for $k = 0$ black holes (single charge case). Our results agree with theirs.

These five dimensional Euclidean R-charged black holes with topology $\mathcal{B}^2 \times \mathbf{S^1} \times \mathbf{R^2}$ are dual to the Euclidean weakly coupled field theory on $\mathbf{S^1} \times \mathbf{S^1} \times \mathbf{R^2}$ with three chemical potentials turned on [6]. The linear behavior of instability lines in phase diagram are also expected in dual gauge theory side. When one turns on some non-zero chemical potential in the gauge theory, this chemical potential acts like a negative mass squared term for the scalars [28]. So if we consider the theory at zero temperature and zero coupling then it does not have any stable ground state, as the potential for the scalar fields is unbounded from bellow. But at some finite temperature the scalars gain a thermal mass at one loop level which is proportional to $\sqrt{\lambda}T$, where λ is 't Hooft coupling. And hence, as long as maximal chemical potential is less than $\sqrt{\lambda}T$ the theory has stable ground state. So in the gauge theory side also we can see linear behavior of instability line.

It would be interesting to find out confinement - deconfinement phase transition on the weak coupling side by computing the partition function of $\mathcal{N} = 4 \ SYM$ theory on $\mathbf{S^1} \times \mathbf{R^2}$ with three nonzero chemical potential [7]. Also one has to keep in mind that the fermions are antiperiodic along the spatial $\mathbf{S^1}$. It has been argued in [30] from the point of view of AdS/CFT that in the limit of large N a conformally invariant gauge theory on a flat torus (with anti-periodic boundary conditions for the fermions in all the compact directions) undergoes a phase transition when two shortest periodicities are interchanged. Therefore, it

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[6]Gauge theory on $\mathbf{R^3}$ or $\mathbf{S^3}$ with $\mathbf{U(1)}$ R-charges has been discussed in [27, 28].
[7]see [29] for related discussion.

seems to be interesting to write a gauge invariant partition function for the gauge theory on $\mathbf{S^1} \times \mathbf{R^2}$ in presence of (three) chemical potential and understand how size of the $\mathbf{S^1}$ circle governs confinement-deconfinement phase transition in the weak coupling side.

Appendix

A Calculation of Free energy for R-charged Black Holes

In order to calculate the on-shell action we will write the action 3.1 using equations of motion as,

$$I_B^{bulk} = -\frac{1}{8\pi G_5} \int d^5x \sqrt{-g} R_\theta^\theta. \tag{A.1}$$

Given the metric in the form,

$$ds^2 = -e^{-4A(r)} f(r) dt^2 + e^{2B(r)} \left(\frac{dr^2}{f(r)} + \frac{r^2}{b^2}(d\theta^2 + d\vec{x}^2) \right), \tag{A.2}$$

the action A.1 becomes,

$$I_B^{bulk} = \frac{\beta V \eta}{8\pi G_5} \left(\frac{\tilde{R}^2 f(\tilde{R})}{b^3} + \frac{\tilde{R}^3 f(\tilde{R}) A'(\tilde{R})}{b^3} \right). \tag{A.3}$$

We have to add the following GH term to the action 3.1,

$$I_B^{GH} = \frac{1}{8\pi G_5} \int d^4x \sqrt{-h} \Theta, \tag{A.4}$$

where h is induced metric on the boundary, $\Theta = -\nabla_\mu n^\mu$ and n is unit normal in the r direction, $n_r = \sqrt{g_{rr}}$. Using metric A.2, the GH term becomes,

$$I_B^{GH} = -\frac{\beta V \eta}{8\pi G_5} \left(\frac{\tilde{R}^2(\tilde{R} f'(\tilde{R}) + 2f(\tilde{R})(3 + \tilde{R} A'(\tilde{R})))}{2b^3} \right). \tag{A.5}$$

Hence the on-shell black hole action is given by,

$$I_B = -\frac{\beta V \eta}{8\pi G_5} \left(\frac{\tilde{R}^2(4f(\tilde{R}) + \tilde{R} f'(\tilde{R}))}{2b^3} \right). \tag{A.6}$$

The function $f(r)$ is given by,

$$f(r) = -\frac{M}{r^2} + \frac{r^2}{b^2} H_1 H_2 H_3 \tag{A.7}$$

and

$$H_I = 1 + \frac{q_I}{r^2}. \tag{A.8}$$

Putting $f(r)$ and $H(r)$ in equation A.6 we get (in $b = 1$ unit),

$$I_B = -\frac{\beta V \eta}{8\pi G_5} \left[-M + (q_1 q_2 + q_2 q_3 + q_3 q_1) + 2\tilde{R}^2(q_1 + q_2 + q_3) + 3\tilde{R}^4 \right]. \tag{A.9}$$

Now we will compute the soliton action on equation of motion following the same steps as above. Soliton metric is given by 3.38. Using that, the bulk soliton action is given by,

$$I_S^{bulk} = \frac{\beta_S V \eta_S}{8\pi G_5} \left[\tilde{R}^4 \left((H_1 H_2 H_3)^{2/3} - \frac{\mu^4}{\tilde{R}^4} \right) \right] \tag{A.10}$$

and the GH boundary action is given by,

$$I_S^{GH} = -\frac{\beta_S V \eta_S}{8\pi G_5} \left[4\tilde{R}^4 (H_1 H_2 H_3)^{2/3} \left(1 - \frac{\mu^4}{2\tilde{R}^4 (H_1 H_2 H_3)^{2/3}} \right) \right]. \tag{A.11}$$

Hence the on-shell soliton action is given by,

$$\begin{aligned} I_S &= I_S^{bulk} + I_S^{GH} \\ &= \frac{\beta_S V \eta_s}{8\pi G_5} \left(-3\tilde{R}^4 (H_1 H_2 H_3)^{2/3} + \mu^4 \right). \end{aligned} \tag{A.12}$$

Using the relation between (β, η) and (β_S, η_S) we find the subtracted action is,

$$I = -\frac{\beta V \eta_S}{16\pi G_5} \left(M + \frac{2}{3}(q_1^2 + q_2^2 + q_3^2) - \frac{2}{3}(q_1 q_2 + q_2 q_3 + q_3 q_1) - \mu^4 \right). \tag{A.13}$$

References

[1] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[2] D. J. Gross and E. Witten, "Possible Third Order Phase Transition In The Large N Lattice Gauge Theory," Phys. Rev. D **21**, 446 (1980).

[3] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[4] S. W. Hawking and D. N. Page, "Thermodynamics Of Black Holes In Anti-De Sitter Space," Commun. Math. Phys. **87**, 577 (1983).

[5] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," Adv. Theor. Math. Phys. **2**, 505 (1998) [arXiv:hep-th/9803131].

[6] J. P. S. Lemos, Black Hole In General Relativity," Phys. Lett. B **353**, 46 (1995) [arXiv:gr-qc/9404041]; J. P. Lemos, Class. Quant. Grav. **12**, 1081 (1995)

[gr-qc/9407024]; J. P. S. Lemos and V. T. Zanchin, Phys. Rev. D **54**, 3840 (1996) [arXiv:hep-th/9511188]. C. G. Huang and C. B. Liang, Phys. Lett. A **201** (1995) 27; R. G. Cai and Y. Z. Zhang, Phys. Rev. D **54**, 4891 (1996) [arXiv:gr-qc/9609065]; R. G. Cai, J. Y. Ji and K. S. Soh, Phys. Rev. D **57**, 6547 (1998) [arXiv:gr-qc/9708063]. R. G. Cai, Nucl. Phys. B **524**, 639 (1998) [arXiv:gr-qc/9801098]; S. Aminneborg, I. Bengtsson, S. Holst and P. Peldan, Class. Quant. Grav. **13**, 2707 (1996) [gr-qc/9604005]. R. B. Mann, Class. Quant. Grav. **14**, L109 (1997) [gr-qc/9607071]; R. B. Mann, Nucl. Phys. B **516**, 357 (1998) [hep-th/9705223]. D. Klemm, Class. Quant. Grav. **15**, 3195 (1998) [gr-qc/9808051]; D. Klemm, V. Moretti and L. Vanzo, Phys. Rev. D **57**, 6127 (1998) [Erratum-ibid. D **60**, 109902 (1998)] [gr-qc/9710123]; M. Banados, A. Gomberoff and C. Martinez, Class. Quant. Grav. **15**, 3575 (1998) [hep-th/9805087]. M. F. A. da Silva, A. Wang, F. M. Paiva and N. O. Santos, Phys. Rev. D **61**, 044003 (2000) [arXiv:gr-qc/9911013]; R. G. Cai, Phys. Lett. B **572**, 75 (2003) [arXiv:hep-th/0306140]; R. Aros, R. Troncoso and J. Zanelli, Phys. Rev. D **63**, 084015 (2001) [hep-th/0011097]. R. G. Cai, Phys. Rev. D **65**, 084014 (2002) [arXiv:hep-th/0109133]. R. G. Cai, Phys. Lett. B **582**, 237 (2004) [arXiv:hep-th/0311240]; W. L. Smith and R. B. Mann, Phys. Rev. D **56**, 4942 (1997) [gr-qc/9703007]; Y. Wu, M. F. A. da Silva, N. O. Santos and A. Wang, Phys. Rev. D **68**, 084012 (2003) [arXiv:gr-qc/0309002]; M. Cvetic, S. Nojiri and S. D. Odintsov, Nucl. Phys. B **628**, 295 (2002) [arXiv:hep-th/0112045]. S. Nojiri and S. D. Odintsov, Phys. Rev. D **66**, 044012 (2002) [arXiv:hep-th/0204112]. Y. M. Cho and I. P. Neupane, Phys. Rev. D **66**, 024044 (2002) [arXiv:hep-th/0202140]. I. P. Neupane, Phys. Rev. D **67**, 061501 (2003) [arXiv:hep-th/0212092]. I. P. Neupane, arXiv:hep-th/0302132; and references therein.

[7] L. Vanzo, "Black holes with unusual topology," Phys. Rev. D **56**, 6475 (1997) [arXiv:gr-qc/9705004].

[8] D. Birmingham, "Topological black holes in anti-de Sitter space," Class. Quant. Grav. **16**, 1197 (1999) [arXiv:hep-th/9808032].

[9] S. Dutta and R. Gopakumar, "On Euclidean and noetherian entropies in AdS space," Phys. Rev. D **74**, 044007 (2006) [arXiv:hep-th/0604070].

[10] Y. S. Myung, "Phase transitions for the topological AdS-black holes and de Sitter spaces," Phys. Lett. B **645**, 369 (2007) [arXiv:hep-th/0603200].

[11] G. Koutsoumbas, S. Musiri, E. Papantonopoulos and G. Siopsis, "Quasi-normal modes of electromagnetic perturbations of four-dimensional topological black holes with scalar hair," JHEP **0610**, 006 (2006) [arXiv:hep-th/0606096].

[12] S. Surya, K. Schleich and D. M. Witt, "Phase transitions for flat adS black holes," Phys. Rev. Lett. **86**, 5231 (2001) [arXiv:hep-th/0101134].

[13] G. T. Horowitz and R. C. Myers, "The AdS/CFT correspondence and a new positive energy conjecture for general relativity," Phys. Rev. D **59**, 026005 (1999) [arXiv:hep-th/9808079].

[14] S. Nojiri and S. D. Odintsov, "Anti-de Sitter black hole thermodynamics in higher derivative gravity and new confining-deconfining phases in dual CFT," Phys. Lett. B **521**, 87 (2001) [Erratum-ibid. B **542**, 301 (2002)] [arXiv:hep-th/0109122].

[15] R. G. Cai, S. P. Kim and B. Wang, "Ricci Flat Black Holes and Hawking-Page Phase Transition in Gauss-Bonnet Gravity and Dilaton Gravity," arXiv:0705.2469 [hep-th].

[16] T. Nishioka and T. Takayanagi, "AdS bubbles, entropy and closed string tachyons," JHEP **0701**, 090 (2007) [arXiv:hep-th/0611035].

[17] N. D. Birrell and P. C. W. Davies, "Quantum Fields In Curved Space," *Cambridge, Uk: Univ. Pr. (1982) 340p*

[18] R. C. Myers, "Stress tensors and Casimir energies in the AdS/CFT correspondence," Phys. Rev. D **60**, 046002 (1999) [arXiv:hep-th/9903203].

[19] M. Cvetic and S. S. Gubser, "Thermodynamic stability and phases of general spinning branes," JHEP **9907**, 010 (1999) [arXiv:hep-th/9903132].

[20] A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, "Charged AdS black holes and catastrophic holography," Phys. Rev. D **60**, 064018 (1999) [arXiv:hep-th/9902170].

A. Chamblin, R. Emparan, C. V. Johnson and R. C. Myers, "Holography, thermodynamics and fluctuations of charged AdS black holes," Phys. Rev. D **60**, 104026 (1999) [arXiv:hep-th/9904197].

[21] R. G. Cai and A. z. Wang, "Thermodynamics and stability of hyperbolic charged black holes," Phys. Rev. D **70**, 064013 (2004) [arXiv:hep-th/0406057].

[22] M. Cvetic *et al.*, "Embedding AdS black holes in ten and eleven dimensions," Nucl. Phys. B **558**, 96 (1999) [arXiv:hep-th/9903214].

[23] T. Torii and H. Maeda, Phys. Rev. D **72**, 064007 (2005) [arXiv:hep-th/0504141].

[24] A. Buchel and L. A. Pando Zayas, "Hagedorn vs. Hawking-Page transition in string theory," Phys. Rev. D **68**, 066012 (2003) [arXiv:hep-th/0305179].

[25] J. T. Liu and W. A. Sabra, "Mass in anti-de Sitter spaces," Phys. Rev. D **72**, 064021 (2005) [arXiv:hep-th/0405171].

[26] M. Cvetic and S. S. Gubser, "Phases of R-charged black holes, spinning branes and strongly coupled gauge theories," JHEP **9904**, 024 (1999) [arXiv:hep-th/9902195].

[27] S. S. Gubser, "Thermodynamics of spinning D3-branes," Nucl. Phys. B **551**, 667 (1999) [arXiv:hep-th/9810225].

[28] P. Basu and S. R. Wadia, "R-charged AdS(5) black holes and large N unitary matrix models," Phys. Rev. D **73**, 045022 (2006) [arXiv:hep-th/0506203].

D. Yamada and L. G. Yaffe, "Phase diagram of N = 4 super-Yang-Mills theory with R-symmetry chemical potentials," JHEP **0609**, 027 (2006) [arXiv:hep-th/0602074].

[29] O. Aharony, J. Marsano, S. Minwalla, K. Papadodimas, M. Van Raamsdonk and T. Wiseman, "The phase structure of low dimensional large N gauge theories on tori," JHEP **0601**, 140 (2006) [arXiv:hep-th/0508077].

[30] D. N. Page, "Phase transitions for gauge theories on tori from the AdS/CFT correspondence," arXiv:hep-th/0205001.